Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 5, pp. 740–753, 2005

SOLVING PROBLEMS OF ELASTICITY OF CONICAL SHELLS

B. V. Nerubailo and L. G. Smirnov

UDC 539.3

The problem of a local action of normal pressure on a thin circular shell is solved by the method of asymptotic synthesis. A conical shell under the action of local radial loads is considered as an example.

Key words: *method of analytical synthesis, conical shell, local action.*

Introduction. In the method of asymptotic synthesis of the stress state [1], based on "matching" of solutions of approximate equations [2], the stress–strain state of a shell is presented as the sum of the principal state, simple edge effect, and bending stress state. These states are described by differential equations with variable coefficients of lower order and a simpler structure than the initial system of differential equations of the general theory of conical shells [3]. The problem reduces to linear differential equations with polynomial coefficients; hence, it is convenient to seek for the solution of the corresponding homogeneous equations in the form of power series. In the case of the bending state, independent solutions of the homogeneous equation are simple power functions. If all independent solutions of the homogeneous equations are known, one can use the Green function to construct particular solutions corresponding to various force actions reflected in the right side of the problem equation. To calculate the principal state, the particular solution is sought in the form of power series by means of expansion of the right side into a power series with a prior approximation of the Heaviside functions and their derivatives in the right side by power series with an approximate representation of the delta function. The technique applied to calculate the principal state can also be used for the bending state and edge effect. If the solutions of the homogeneous equation are known, the Green function for linear differential equations can be readily found by simple procedures, which was taken into account in calculating the edge effect in the present work. In the case of the bending state, the Mellin transform allows one to obtain a simple particular analytical solution. The Mellin transform is also used to calculate the edge effect.

1. Formulation of the Problem. We consider a conical shell with the following elastic and geometric characteristics: Young's modulus E, Poisson's ratio ν, shell thickness h, angle between the meridional line and cone centerline θ, coordinate along the meridional line r, circular coordinate β, and r-coordinates (r_0 and R_0) corresponding to the end sections of the shell ($r_0 < R_0$); the normal pressure acts on the interval $r_1 \leqslant r \leqslant r_2$.

It is assumed that piecewise-constant normal pressure q_0 acts on k rectangular domains in the coordinate system (r, β); these domains are uniformly distributed along the shell contour (Fig. 1).

The load $q(r, \beta)$ can be represented as a series

$$q(r,\beta) = q(r) \sum_{n=0}^{\infty} \varphi_n \cos(kn\beta),$$

where $q(r) = q_0\varphi(r)$, $\varphi(r) = \Omega(r - r_1) - \Omega(r - r_2)$, $\varphi_0 = k\beta_0/\pi$, $\varphi_n = (2/\pi n)\sin(kn\beta_0)$ ($n = 1, 2, 3, \ldots, \infty$, $k = 1, 2, 3, \ldots$, $k\beta_0 \leqslant \pi$), and $\Omega(r - r_1)$ and $\Omega(r - r_2)$ are the Heaviside functions.

Note that the contribution of each elementary stress state can be different in determining a particular force factor and displacement. Thus, for $n < n^*$ (n^* is a certain integer), the longitudinal and shear forces are determined

Moscow State Aviation Institute (Technical University), Moscow 125871; prof_nebo@mail.ru, borisn@km.ru.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 5, pp. 150–165, September–October, 2005. Original article submitted August 26, 2003; revision submitted November 30, 2004.

 0021-8944/05/4605-0740

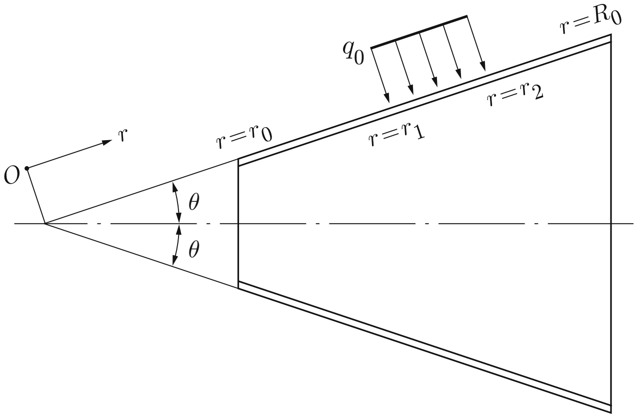

Fig. 1

by the principal state, the circumferential force is determined by the edge effect, and the bending moments and normal displacements are determined by both factors [4].

2. Principal State of the Shell. If we use the circumferential displacement $V(r,\beta)$ as the resolving function of the principal state and expand it into a series in terms of the circumferential coordinate β as

$$V(r,\beta)=\sum_{n=1}^{n^*} V_n(r)\sin(kn\beta),$$

we can write a differential equation for the amplitudes $V_n(r)$ [5]

$$\frac{d^2}{dr^2}\Big\{r\Big[\tan^2\theta+12(1-\nu^2)\,\frac{r^2}{h^2}\Big]\frac{d^2V_n}{dr^2}\Big\}-2\mu_n^2\tan\theta\Big[\frac{d^2V_n}{rdr^2}+\frac{d^2}{dr^2}\Big(\frac{V_n}{r}\Big)\Big]+4\mu_n^4\,\frac{V_n}{r^3}$$

$$=\frac{q_0}{D}\,\varphi_n kn\Big\{\frac{d}{dr}\,(r^2\varphi(r))-\Big(\frac{k^2n^2}{\sin^2\theta}-1\Big)r\varphi(r)\Big\}, \tag{1}$$

where

$$\mu_n^4=\frac{k^4n^4(k^2n^2-1)^2}{4\sin^6\theta\cos^2\theta};\qquad V_n=V_n(r);\qquad D=\frac{Eh^3}{12(1-\nu^2)}.$$

We introduce the following notation: $a_1=\tan^2\theta$, $a_2=12(1-\nu^2)/h^2$, $a_3=2\mu_n^2\tan\theta$, and $a_4=4\mu_n^4$. Then, Eq. (1) acquires the form

$$\frac{d^2}{dr^2}\Big\{r(a_1+a_2r^2)\,\frac{d^2V_n}{dr^2}\Big\}-a_3\Big\{\frac{d^2V_n}{r\,dr^2}+\frac{d^2}{dr^2}\Big(\frac{V_n}{r}\Big)\Big\}+a_4\,\frac{V_n}{r^3}=f(r). \tag{2}$$

The right side of Eq. (1) is designated in Eq. (2) by $f(r)$.

We seek for the solution of Eq. (2) in the form

$$V_n(r)=\tilde V_n(r)+\hat V_n(r), \tag{3}$$

where $\tilde V_n(r)$ is the solution of a homogeneous equation corresponding to Eq. (2) and $\hat V_n(r)$ is some particular solution of Eq. (2).

The solution of the homogeneous equation (since it is linear and has the fourth order) is represented as the sum

$$\tilde V_n(r)=\sum_{s=1}^{4} C_s f_s(r), \tag{4}$$

where $f_s(r)$ are some linearly independent solutions of the homogeneous equation and C_s are arbitrary constants $(s=1,2,3,4)$. We seek for the functions $f_s(r)$ in the form of the series

$$f_s(r) = \sum_{m=1}^{\infty} b_m^{(s)} (r-a)^m = \sum_{m=0}^{\infty} b_m^{(s)} \xi^m = f_s^*(\xi), \tag{5}$$

where a is a certain positive number ($a > r_0$).

Thus, with the use of the variable $\xi = r - a$, the homogeneous equation corresponding to Eq. (2) can be written in the form

$$[a_1(\xi+a)^4 + a_2(\xi+a)^6] \frac{d^4V_n^*}{d\xi^4} + 2[a_1(\xi+a)^3 + 3a_2(\xi+a)^5] \frac{d^3V_n^*}{d\xi^3}$$

$$+ [6a_2(\xi+a)^4 - 2a_3(\xi+a)^2] \frac{d^2V_n^*}{d\xi^2} + 2a_3(\xi+a) \frac{dV_n^*}{d\xi} - (2a_3 - a_4)V_n^* = 0 \tag{6}$$

or

$$P_4(\xi) \frac{d^4V_n^*}{d\xi^4} + P_3(\xi) \frac{d^3V_n^*}{d\xi^3} + P_2(\xi) \frac{d^2V_n^*}{d\xi^2} + P_1(\xi) \frac{dV_n^*}{d\xi} + P_0(\xi)V_n^* = 0, \tag{7}$$

where $P_4(\xi) = a_2\xi^6 + 6aa_2\xi^5 + (a_1 + 15a^2a_2)\xi^4 + (4aa_1 + 20a^3a_2)\xi^3 + (6a^2a_1 + 15a^4a_2)\xi^2 + (4a^3a_1 + 6a^5a_2)\xi + a^4a_1 + a^6a_2$, $P_3(\xi) = 6a_2\xi^5 + 30aa_2\xi^4 + (2a_1 + 60a^2a_2)\xi^3 + (6aa_1 + 60a^3a_2)\xi^2 + (6a^2a_1 + 30a^4a_2)\xi + 2a^3a_1 + 6a^5a_2$, $P_2(\xi) = 6a_2\xi^4 + 24aa_2\xi^3 + (36a^2a_2 - 2a_3)\xi^2 + (24a^3a_2 - 4aa_3)\xi + 6a^4a_2 - 2a^2a_3$, $P_1(\xi) = 2a_3\xi + 2aa_3$, and $P_0(\xi) = -2a_3 + a_4$.

Substituting expression (5) into (6) and (7) and equating the coefficients at ξ^m to zero, we obtain the relation

$$A_m^{(1)} b_{m-2} + A_m^{(2)} b_{m-1} + A_m^{(3)} b_m + A_m^{(4)} b_{m+1} + A_m^{(5)} b_{m+2} + A_m^{(6)} b_{m+3} + A_m^{(7)} b_{m+4} = 0, \tag{8}$$

where

$$\begin{gathered}
A_m^{(1)} = (m-1)(m-2)^2(m-3)a_2, \qquad A_m^{(2)} = 6aa_2(m-1)^3(m-2), \\
A_m^{(3)} = a_1 m(m-1)^2(m-2) + 3a^2a_2 m(m-1)(5m^2 - 5m + 2) - 2a_3(m-1)^2 + a_4, \\
A_m^{(4)} = 2aa_1(m-1)m(m+1)(2m+1) + 4a^3a_2 m(m+1)(5m^2+1) - 2a_3(m+1)(2m-1), \\
A_m^{(5)} = 6a^2a_1 m^2(m+1)(m+2) + 3a^4a_2(m+1)(m+2)(5m^2+5m+2) - 2a^2a_3(m+1)(m+2), \\
A_m^{(6)} = 2a^3a_1(m+1)(m+2)(m+3)(2m+1) + 6a^5a_2(m+1)^2(m+2)(m+3), \\
A_m^{(7)} = (a^4a_1 + a^6a_2)(m+1)(m+2)(m+3)(m+4).
\end{gathered} \tag{9}$$

We write Eq. (8) in the form

$$b_{m+4} = -[A_m^{(6)} b_{m+3} + A_m^{(5)} b_{m+2} + A_m^{(4)} b_{m+1} + A_m^{(3)} b_m + A_m^{(2)} b_{m-1} + A_m^{(1)} b_{m-2}]/A_m^{(7)}. \tag{10}$$

As $A_m^{(7)} = 0$ for $m = -1$, -2, -3, and -4 ($b_{-1} = b_{-2} = 0$), assuming that $b_0^{(1)} = C_1$, $b_1^{(2)} = C_2$, $b_2^{(3)} = C_3$, and $b_3^{(4)} = C_4$ [C_s are constants in Eq. (4)], we can use formulas (9) and (10) to obtain four sequences of the coefficients $b_m^{(s)}$ ($s = 1, 2, 3, 4$; $m = m_s, m_s + 1, \ldots, \infty$), which determine four functions $f_s(r)$ — independent solutions of the homogeneous equation, which follows from their construction.

Thus, the general solution of the homogeneous equation has the form

$$\tilde{V}_n(r) = \tilde{V}_n^*(\xi) = \sum_{s=1}^{4} C_s f_s(\xi).$$

To find some particular solution of the equation

$$P_4(\xi) \frac{d^4V_n^*}{d\xi^4} + P_3(\xi) \frac{d^3V_n^*}{d\xi^3} + P_2(\xi) \frac{d^2V_n^*}{d\xi^2} + P_1(\xi) \frac{dV_n^*}{d\xi} + P_0(\xi)V_n^* = f^*(\xi), \tag{11}$$

where

$$f^*(\xi) = B_n \Big\{ \frac{d}{d\xi} [(\xi+a)^2 \varphi(\xi+a)] - \Big(\frac{k^2n^2}{\sin^2\theta} - 1 \Big)(\xi+a)\varphi(\xi+a) \Big\}; \qquad B_n = \frac{q_0}{D}\, kn\varphi_n,$$

we use the approximate representation for the δ function [6]

$$\delta(\xi-\xi_0)\approx\sqrt{\alpha/\pi}\,\exp\left(-\alpha(\xi-\xi_0)^2\right)=\bar{\delta}(\xi-\xi_0) \tag{12}$$

(α is a certain positive quantity [6]).

The expansion for $\delta(\xi-\xi_0)$ in (12) can be presented as

$$\delta(\xi-\xi_0)=\sqrt{\alpha/\pi}\,[1-\alpha(\xi-\xi_0)^2/1!+\alpha^2(\xi-\xi_0)^4/2!-\dots]. \tag{13}$$

The function $f(r)=f^*(\xi)$ can be written as

$$f(r)=f^*(\xi)=B_n\{(k^2n^2/\sin^2\theta-\xi)(\xi+a)[\Omega(\xi-\xi_2)-\Omega(\xi-\xi_1)$$
$$+(\xi+a)^2[\delta(\xi-\xi_1)-\delta(\xi-\xi_2)]]\}\qquad(\xi_j=r-r_j;\quad j=1,2).$$

For the approximate representation of the Heaviside function, Eq. (13) yields

$$\Omega(\xi-\xi_0)=\int\limits_{-\infty}^{\xi}\delta(\zeta-\zeta_0)\,d\zeta=\sqrt{\frac{\alpha}{\pi}}\left[(\xi-\xi_0)-\frac{\alpha(\xi-\xi_0)^3}{3\cdot 1!}+\frac{\alpha^2(\xi-\xi_0)^5}{5\cdot 2!}-\dots\right]$$
$$=\sqrt{\frac{\alpha}{\pi}}\sum_{n=0}^{\infty}\frac{(-\alpha)^n(\xi-\xi_0)^{2n+1}}{(2n+1)n!}. \tag{14}$$

Expanding $f^*(\xi)$ into a Taylor series, with allowance for Eqs. (13) and (14), we obtain

$$f^*(\xi)=\sum_{m=1}^{\infty}\frac{d_m}{m!}\,\xi^m,\qquad d_m=d_m^{(1)}+d_m^{(2)}+d_m^{(3)}+d_m^{(4)}+d_m^{(5)},$$

where

$$d_m^{(1)}=\sqrt{\frac{\alpha}{\pi}}\,B_n\Big(\frac{k^2n^2}{\sin^2\theta}-3\Big)a\sum_{j=0}^{\infty}\frac{D_m^{(1)}[(-\xi_2)^{2j+1-m}-(-\xi_1)^{2j+1-m}](-\alpha)^j}{(2j+1)j!};$$

$$d_m^{(2)}=\sqrt{\frac{\alpha}{\pi}}\,B_n\Big(\frac{k^2n^2}{\sin^2\theta}-3\Big)\sum_{j=0}^{\infty}\frac{D_m^{(2)}[(-\xi_2)^{2j+2-m}-(-\xi_1)^{2j+2-m}](-\alpha)^j}{j!(2j+1)};$$

$$d_m^{(3)}=\sqrt{\frac{\alpha}{\pi}}\,a^2\sum_{j=0}^{\infty}\frac{D_m^{(3)}[(-\xi_1)^{2j-m}-(-\xi_2)^{2j-m}](-\alpha)^j}{j!};$$

$$d_m^{(4)}=2\sqrt{\frac{\alpha}{\pi}}\,a\sum_{j=0}^{\infty}\frac{D_m^{(4)}[(-\xi_1)^{2j-m+1}-(-\xi_2)^{2j-m+1}](-\alpha)^j}{j!};$$

$$d_m^{(5)}=\sqrt{\frac{\alpha}{\pi}}\sum_{j=0}^{\infty}\frac{D_m^{(5)}[(-\xi_1)^{2j-m+2}-(-\xi_2)^{2j-m+2}]}{j!};$$

$$D_m^{(1)}=D_m^{(2)}=D_m^{(3)}=D_m^{(4)}=D_m^{(5)}=1\qquad\text{at}\quad m=0;$$

$$D_m^{(1)}=(2j+1)2j(2j-1)\cdots(2j-m+1),$$

$$D_m^{(2)}=(2j+1)2j(2j-1)\cdots(2j-m+2),$$

$$D_m^{(3)}=2j(2j-1)\cdots(2j-m+1),\qquad D_m^{(4)}=2j(2j-1)\cdots(2j-m+2),$$

$$D_m^{(5)}=2j(2j-1)\cdots(2j-m+3)\qquad\text{for}\quad m\geqslant 1.$$

Substituting now the expansion for the function $f^*(\xi)$ into Eq. (11) and the expansion for the particular solution of the form

$$\hat{V}_n^*(\xi) = \sum_{m=0}^{\infty} C'_m \xi^m$$

and equating the coefficients at ξ^m in the right and left sides of the new equality, we obtain equations for determining the unknown coefficients C'_m:

$$A_m^{(1)} C'_{m-2} + A_m^{(2)} C'_{m-1} + A_m^{(3)} C'_m + A_m^{(4)} C'_{m+1} + A_m^{(5)} C'_{m+2} A_m^{(6)} C'_{m+3} + A_m^{(7)} C'_{m+4} = d'_m \tag{15}$$

$$(m = 0, 1, 2, \ldots; \qquad C'_{-1} = C'_{-2} = 0; \qquad d'_m = d_m/m!).$$

Assuming that $C'_0 = C'_1 = C'_2 = C'_3 = 0$, we can obtain one of the particular solutions of the infinite system of equations (15) in a recurrent manner from relations (15).

Thus, with accuracy to arbitrary constants, we found the solution [see Eq. (3)]

$$V_n(r) = \sum_{j=1}^{4} C_j \tilde{V}_{nj}(r) + \hat{V}_n(r),$$

in which C_j are unknown constants and $\tilde{V}_{nj}(r)$ are independent solutions of the homogeneous equation ($j = 1, 2, 3, 4$).

The displacements, normal forces, and bending moments are determined by the following formulas:

$$V(r, \beta) = \sum_{n}^{n^*} V_n(r) \sin(kn\beta),$$

$$u(r, \beta) = \frac{\sin\theta}{k} \sum_{n}^{n^*} \frac{1}{n} \Big(r \frac{dV_n}{dr} - V_n \Big) \cos(kn\beta),$$

$$w(r, \beta) = \sum_{n}^{n^*} \Big[\Big(\frac{\sin^2\theta}{kn} - kn \Big) V_n - \frac{\sin^2\theta}{kn} r \frac{dV_n}{dr} \Big] \cos(kn\beta),$$

$$T_1(r, \beta) = Eh \frac{\sin\theta}{k} \sum_{n}^{n^*} \frac{r}{n} \frac{d^2 V_n}{dr^2} \cos(kn\beta), \tag{16}$$

$$G_1(r, \beta) = \frac{D}{\cos\theta} \sum_{n}^{n^*} \Big[\Big(kn + \frac{\sin^2\theta}{kn} \Big) \frac{d^2 V_n}{dr^2} + \frac{r \sin^2\theta}{kn} \frac{d^3 V_n}{dr^3} \Big] \cos(kn\beta),$$

$$G_2(r, \beta) = \frac{D}{\cos\theta} \sum_{n}^{n^*} \Big[\frac{\sin^2\theta}{kn} \frac{d^2 V_n}{dr^2} - \frac{kn(k^2 n^2 - 1)}{r^2 \sin^2\theta} \Big] \cos(kn\beta);$$

summation in (16) is performed from $n = 1$ for $k \geqslant 2$ and from $n = 2$ for $k = 1$ to $n = n^*$ [4].

To find the unknown constants C_s ($s = 1, 2, 3, 4$) in (4), we have to use four boundary conditions, which can have different forms. Substituting $V(r, \beta)$ into the boundary conditions, which are linear in terms of r, we obtain

$$\sum_{s=1}^{4} C_s \bar{L}_1[\tilde{V}_{nj}(r_0)] + \bar{L}_1[\hat{V}_n(r_0)] = 0, \qquad \sum_{s=1}^{4} C_s \bar{L}_2[\tilde{V}_{nj}(r_0)] + \bar{L}_2[\hat{V}_n(r_0)] = 0,$$

$$\sum_{s=1}^{4} C_s \bar{L}_3[\tilde{V}_{nj}(R_0)] + \bar{L}_3[\hat{V}_n(R_0)] = 0, \qquad \sum_{s=1}^{4} C_s \bar{L}_4[\tilde{V}_{nj}(R_0)] + \bar{L}_4[\hat{V}_n(R_0)] = 0. \tag{17}$$

Here $\bar{L}_1$ and $\bar{L}_2$ are the operators of the boundary conditions for $r = r_0$; $\bar{L}_3$ and $\bar{L}_4$ are the operators of the boundary conditions for $r = R_0$. Conditions (17) are a system of linear algebraic equations with respect to C_s.

Concerning the convergence of series (5), it should be noted that their convergence on the interval (r_0, R_0) (shell boundaries) is observed at least for all $a > r_0$, which directly follows from Eqs. (9) and (10).

3. Edge Effect. The differential equation for the edge effect under the action of the radial load $q(r, \beta)$ has the form [5]

$$\frac{d}{dr}\Big\{r\,\frac{d}{dr}\Big[\frac{1}{r}\,\frac{d}{dr}\Big(r\,\frac{dw}{dr}\Big)\Big]\Big\} + \frac{12(1-\nu^2)}{h^2 r \tan^2\theta}\,w = \frac{r}{D}\,\varphi_n q(r). \tag{18}$$

As in the case of the principal state, we write the solution of Eq. (18) as the sum of the solution of the homogeneous equation and the particular solution:

$$w(r) = \tilde{w}(r) + \hat{w}(r).$$

The homogeneous equation corresponding to (18) can be rewritten as

$$r^3\,\frac{d^4w}{dr^4} + 2r^2\,\frac{d^3w}{dr^3} - r\,\frac{d^2w}{dr^2} + \frac{dw}{dr} + \eta r w = 0, \qquad \eta = \frac{12(1-\nu^2)}{h^2\tan^2\theta}. \tag{19}$$

We seek for the solution $\tilde{w}(r)$ in the form of the series

$$\tilde{w}(r) = \sum_{n=0}^{\infty} a_n (r-a)^n = \sum_{n=0}^{\infty} a_n \xi^n = \tilde{w}^*(\xi), \tag{20}$$

where a is a certain positive number ($a > r_0$).

Substituting $r = \xi + a$ into Eq. (19), we obtain

$$(\xi^3{+}3a\xi^2{+}3a^2\xi{+}a^3)\,\frac{d^4w^*(\xi)}{d\xi^4} + 2(\xi^2+2a\xi+a^2)\,\frac{d^3w^*(\xi)}{d\xi^3} - (\xi+a)\,\frac{d^2w^*(\xi)}{d\xi^2} + \frac{dw^*(\xi)}{d\xi} + \eta(\xi+a)w^*(\xi) = 0. \tag{21}$$

Substituting (20) into (21) and equating the coefficients at ξ^n to zero, we find the relation

$$a^3(n+1)(n+2)(n+3)(n+4)a_{n+4} + a^2(n+1)(n+2)(n+3)(3n+2)a_{n+3}$$
$$+ a(n+1)(n+2)[(3n-1)n-1]a_{n+2} + (n^2-1)^2 a_{n+1} + \eta a a_n + \eta a_{n-1} = 0.$$

Hence, we have

$$a_{n+4} = -\{a^2(n+1)(n+2)(n+3)(3n+2)a_{n+3} + a(n+1)(n+2)[(3n-1)n-1]a_{n+2}$$
$$+ (n^2-1)^2 a_{n+1} + \eta a a_n + \eta a_{n-1}\}/[a^3(n+1)(n+2)(n+3)(n+4)]. \tag{22}$$

As the nominator in the right side of Eq. (22) equal zero for $n = -4$, -3, -2, and -1, the coefficients a_0, a_1, a_2, and a_3 can be assumed to be arbitrary constants: $a_{s-1} = C_s$ ($s = 1$, 2, 3, and 4). Then, the coefficients $a_n^{(s)}$ ($s = 1$, 2, 3, and 4) are found from the recurrent formula (22) with accuracy to an arbitrary set $C_s = \{(C_1,0,0,0),(0,C_2,0,0),(0,0,C_3,0),(0,0,0,C_4)\}$; any element of the set C_s can be used as the coefficient a_0, a_1, a_2, or a_3.

Thus, the resultant solution of the homogeneous equation acquires the form

$$\tilde{w}^*(\xi) = \sum_{s=1}^{4} C_s \tilde{w}_s^*(\xi),$$

where the functions $\tilde{w}_s^*(\xi) = \sum\limits_{n=s-1}^{\infty} a_n^{(s)}\xi^n$ are independent solutions, which follows from their construction.

It follows from Eq. (22) that these series converge at the interval $[r_0, R_0]$, at least for $a > r_0$.

To obtain the solution of the heterogeneous equation, we apply the Mellin transform [7] to both sides of Eq. (18); for this purpose, we use the formulas related to the Mellin transform [8]

$$tg'(t) \div -pg(p), \qquad t^2 g''(t) \div p(p+1)G(p),$$
$$t^3 g'''(t) \div -p(p+1)(p+2)G(p), \qquad t^4 g^{IV}(t) \div p(p+1)(p+2)(p+3)G(p), \tag{23}$$

where $G(p)$ is the Mellin transform of the function $g(t)$:

$$G(p) = \int_0^{\infty} g(t)t^{p-1}\,dt.$$

Passing to the new variable $\xi = r/(\alpha r_0)$, we can write Eq. (18) in the form

$$\xi \frac{d}{d\xi}\Big\{\xi \frac{d}{d\xi}\Big[\frac{d}{\xi d\xi}\Big[\xi \frac{dw^*(\xi)}{d\xi}\Big]\Big]\Big\} + w^*(\xi) = \gamma \xi^2 q^*(\xi), \tag{24}$$

where $\alpha = h\tan\theta/\sqrt{12(1-\nu^2)}$, $\gamma = (h^2\tan^2\theta/(12(1-\nu^2)D)\alpha^2\varphi_n$, $q^*(\xi) = q_0$ for $r_1/(\alpha r_0) \leqslant \xi \leqslant r_2/(\alpha r_0)$, and $q^*(\xi) = 0$ for all other ξ.

Applying the Mellin transform to both sides of Eq. (24) and taking into account Eq. (23), we find

$$p^2(p-2)^2 G(p-2) + G(p) = F(p), \tag{25}$$

where

$$G(p) = \int_0^\infty \hat{w}^*(\xi)\xi^{p-1}\,d\xi; \qquad F(p) = \int_0^\infty \gamma\xi^2 q^*(\xi)\xi^{p-1}\,d\xi = \gamma\int_{\xi_1}^{\xi_2} \xi^{p+1}\,d\xi = \gamma\,\frac{\xi_2^{p+2} - \xi_1^{p+2}}{p+2}.$$

If we use the procedure described in [9, 10], where a factor ε (small positive quantity) is introduced in the second term in the left side of Eq. (25), then, instead of (25), we have

$$p^2(p-2)^2 G(p-2) + \varepsilon G(p) = F(p). \tag{26}$$

Seeking for $G(p)$ in the form of the series

$$G(p) = \sum_0^\infty G_s(p)\varepsilon^s \tag{27}$$

and substituting (27) into equality (26), we find

$$p^2(p-2)G_0(p-2) = F(p), \qquad p^2(p-2)^2 G_s(p-2) + G_{s-1}(p) = 0 \quad (s = 1, 2, 3, \dots). \tag{28}$$

It follows from Eq. (28) that

$$G_0(p) = \frac{F(p+2)}{p^2(p+2)^2}, \qquad G_s(p) = -\frac{G_{s-1}(p+2)}{p^2(p+2)^2}. \tag{29}$$

Now, using equalities (29), we obtain

$$G_s(p) = (-1)^s F(p+2s+2)/A(p,s), \tag{30}$$

where $A(p,s) = p^2\prod_{j=1}^{s}(p+2j)^4(p+2s+2)^2$ $(s = 1, 2, \dots)$ and $A(p,0) = p^2(p+2)^2$.

It follows from Eq. (30) that the functions $G_s(p)$ are holomorphic for $\operatorname{Re} p \leqslant 0$, except for multiple poles at the points $p_j = -2j$ $(j = 1, 2, \dots, s)$, $p_0 = 0$, and $p_{s+1} = -2s-2$.

The inverse Mellin transform can be written as

$$\hat{w}^*(\xi) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} G_s(p)\xi^{-p}\,dp = \frac{(-1)^s}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{F(p+2s+2)\xi^{-p}}{(p+2s+4)\prod_{j=1}^{s}(p+2j)^4(p+2s+2)^2p^2}\,dp,$$

where $\tau = \varepsilon$ is a small positive number; the product in the denominator is omitted for $s = 0$.

The expression $\hat{w}^*(\xi)$ acquires the form

$$\hat{w}^*(\xi) = \hat{w}^{*(2)}(\xi) - \hat{w}^{*(1)}(\xi).$$

Here,

$$\hat{w}^{*(m)}(\xi) = \frac{B_s\xi_m^{2s+4}}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{\exp(p\ln(\xi_m/\xi))}{p^2(p+2s+2)^2(p+2s+4)\chi_s(p)}\,dp, \tag{31}$$

$$B_s = (-1)^s\gamma; \quad \chi_0(p) = 1; \quad \chi_s(p) = \prod_{j=1}^{s}(p+2j)^4 \quad (s = 1, 2, \dots).$$

We can easily see that the integrand in (31) satisfies Jordan's conditions [11]; $\mathrm{Re}\, p < -\tau$ in the left half-plane for $\xi < \xi_m$ and $\mathrm{Re}\, p > -\tau$ in the right half-plane for $\xi > \xi_m$. To calculate the residues in integral (31), we use the formula for the residue of the function $f(z)$, which has a pole of the nth order:

$$\mathrm{res}\, f(z=a) = \frac{1}{(n-1)!} \lim_{z\to a} \Big\{ \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right] \Big\}. \tag{32}$$

Denoting

$$f(z) = \frac{\exp\left(p \ln\left(\xi_m/\xi\right)\right)}{p^2(p+2s+2)^2(p+2s+4)\chi_s(p)},$$

we find, using Eq. (32), the residues at the points p_s. For $p_0 = 0$ (double pole), we obtain

$$b_0 = \mathrm{res}\, f\Big|_{p=0} = \lim_{p\to 0} \frac{d}{dp}\left[p^2 f(p)\right] = \frac{d}{dp}\Big[\frac{\exp\left(p\ln\left(\xi_m/\xi\right)\right)}{(p+2s+4)(p+2s+2)^2\chi_s(p)}\Big]\Big|_{p=0}. \tag{33}$$

We use the logarithmic derivative for calculating derivative (33). If $\psi(p) = \ln f(p)$, then $\psi'(p) = f'(p)/f(p)$, and hence,

$$b_0 = f'(p)\Big|_{p=0} = \left[\psi'(p) f(p)\right]\Big|_{p=0}. \tag{34}$$

In our case, we have

$$\psi(p) = \ln f(p) = p\, \ln\frac{\xi_m}{\xi} - \ln\left(p+2s+4\right) - 2\ln\left(p+2s+2\right) - 4\sum_{j=1}^{s} \ln\left(p+2j\right),$$

$$\psi'(p) = \ln\frac{\xi_m}{\xi} - \frac{1}{p+2s+4} - \frac{2}{p+2s+2} - 4\sum_{j=1}^{s}\frac{1}{p+2j},$$

and, as it follows from (34),

$$b_0 = \frac{1}{(2s+4)(2s+2)^2\chi_s(0)}\Big(\ln\frac{\xi_m}{\xi} - \frac{1}{2s+4} - \frac{2}{2s+2} - 2\sum_{j=1}^{s}\frac{1}{j}\Big).$$

Similarly, for the double pole $p_{s+1} = -2s-2$, we have

$$b_{s+1} = \frac{\exp\left(-(2s+2)\ln\left(\xi_m/\xi\right)\right)}{2(2s+2)^2\chi_s(-2s-2)}\Big(\ln\frac{\xi_m}{\xi} - \frac{1}{2} + \frac{1}{s+1} - 2\sum_{j=1}^{s}\frac{1}{j-s-1}\Big).$$

To calculate the residue in the poles p_s ($s = 1, 2, \dots$), which have the fourth order, we again use the logarithmic derivative

$$b_j = \mathrm{res}\, f\Big|_{p=-2j} = \frac{1}{6}\lim_{p\to -2j}\frac{d^3}{dp^3}\left[(p+2j)^4 f(p)\right] = \Big[\frac{1}{6}\,\frac{d^3}{dp^3}\, f(p)\Big]\Big|_{p=-2j},$$

where

$$f_j(p) = \frac{\exp\left(p\ln\left(\xi_m/\xi\right)\right)}{(p+2s+4)p^2(p+2s+2)^2\chi_s^{(j)}(p)}; \qquad \chi_s^{(j)}(p) = \prod_{i=1}^{s}(p+2i)^4 \quad (i\neq j).$$

If $\psi_j(p) = \ln f_j(p)$, then

$$\psi_j'(p) = f_j'(p)/f_j(p), \qquad f_j'(p) = \psi_j'(p) f_j(p),$$

$$f_j''(p) = \psi_j''(p) f_j(p) + \psi_j'(p) f_j'(p) = \{\psi_j''(p) + [\psi_j'(p)]^2\} f_j(p),$$

$$f_j'''(p) = \{\psi_j'''(p) + 3\psi_j''(p)\psi_j'(p) + [\psi_j'(p)]^3\} f_j(p) \qquad (j = 1, 2, \dots, s).$$

Hence, we have the expression

$$b_j = \frac{1}{6}\, f_j'''(p) = \frac{1}{6}\,\Psi_j(p) f_j(p) = f_j^*(p)\Big|_{p=-2j},$$

where

$$\Psi_j(p) = \psi_j'''(p) + 3\psi_j''(p)\psi_j'(p) + [\psi_j'(p)]^3,$$

$$\psi_j'(p) = \ln\frac{\xi_m}{\xi} - \frac{1}{p+2s+4} - \frac{2}{p} - \frac{2}{p+2s+2} - 4\sum_{i=1}^{s}\frac{1}{p+2i},$$

$$\psi_j''(p) = \frac{1}{(p+2s+4)^2} + \frac{2}{p^2} + \frac{2}{(p+2s+2)^2} + 4\sum_{i=1}^{s}\frac{1}{(p+2i)^2},$$

$$\psi_j'''(p) = -\frac{2}{(p+2s+4)^3} - \frac{4}{p^3} - \frac{4}{(p+2s+2)^2} - 8\sum_{i=1}^{s}\frac{1}{(p+2i)^3} \qquad (i \neq j).$$

Thus, the residues of the function $f(p)$ are found.

As a result, we obtain

$$\hat{w}^{*(m)}(\xi) = B_s\xi^{2s+4}\Big[\xi^*_{s+2}(-2s-4) + f^*_{s+1}(-2s-2) + \sum_{j=1}^{s} f^*_j(-2j)\Big] \tag{35}$$

for $\xi < \xi_m$ ($m = 1, 2$; $s = 1, 2, \ldots$) and

$$\hat{w}^{*(m)}(\xi) = -B_s\xi_m^{2s+4} f^*_0(0)$$

for $\xi > \xi_m$ ($m = 1, 2$; $s = 1, 2, \ldots$). For $s = 0$, the sum in expression (35) is absent. The function $\hat{w}^*(\xi)$ is now found from Eq. (31).

Note that series (27) with $\varepsilon = 1$, which corresponds to the solution of our problem, converges uniformly in the entire plane of the complex variable p; this directly follows from Eq. (30). Therefore, the inverse Mellin transform of series (27) is justified.

Thus, we found the particular solution

$$\hat{w}^*(\xi) = \sum_{s=0}^{\infty} \hat{w}^*_s(\xi),$$

and the general solution has the form

$$w^*(\xi) = \sum_{j=1}^{4} C_j\tilde{w}^*_j(\xi) + \hat{w}^*(\xi). \tag{36}$$

In (36), the constants C_j ($j = 1, 2, 3, 4$) are found from the boundary conditions.

If, for instance, the boundary conditions have the form (the shell edges being free)

$$\begin{aligned} G_1(r) = -D\Big(\frac{d^2w}{dr^2} + \nu\,\frac{1}{r}\,\frac{dw}{dr}\Big)\Big|_{\substack{r=r_0\\ r=R_0}} &= -D\Big(\frac{d^2w^*}{d\xi^2} + \nu\,\frac{1}{\xi}\,\frac{dw^*}{d\xi}\Big)\Big|_{\substack{\xi=1\\ \xi=R_0/r_0}} = 0,\\ Q_1(r) = -D\Big(\frac{d^3w}{dr^3} + \frac{1}{r}\,\frac{d^2w}{dr^2} - \frac{1}{r^2}\,\frac{dw}{dr}\Big)\Big|_{\substack{r=r_0\\ r=R_0}} &= -D\Big(\frac{d^3w^*}{d\xi^3} + \frac{1}{\xi}\,\frac{d^2w^*}{dr^2} - \frac{1}{\xi^2}\,\frac{dw^*}{d\xi}\Big)\Big|_{\substack{\xi=1\\ \xi=R_0/r_0}} = 0, \end{aligned} \tag{37}$$

then, substituting Eq. (36) into Eq. (37), we obtain a system of linear algebraic equations with respect to C_j ($j = 1, 2, 3, 4$):

$$\sum_{j=1}^{4}\Big(\frac{d^2\tilde{w}^*_j(\xi)}{d\xi^2} + \nu\,\frac{1}{\xi}\,\frac{d\tilde{w}^*_j(\xi)}{d\xi}\Big)C_j + \Big(\frac{d^2\hat{w}^*(\xi)}{d\xi^2} + \nu\,\frac{1}{\xi}\,\frac{d\hat{w}^*(\xi)}{d\xi}\Big) = 0$$

$$(\xi = 1, \qquad \xi = R_0/r_0),$$

$$\sum_{j=1}^{4}\Big(\frac{d^3\tilde{w}^*_j(\xi)}{d\xi^3} + \frac{1}{\xi}\,\frac{d^2\tilde{w}^*_j(\xi)}{d\xi^2} - \frac{1}{\xi^2}\,\frac{d\tilde{w}^*_j(\xi)}{d\xi}\Big)C_j + \Big(\frac{d^3\hat{w}^*_j(\xi)}{d\xi^3} + \frac{1}{\xi}\,\frac{d^2\hat{w}^*_j(\xi)}{d\xi^2} - \frac{1}{\xi^2}\,\frac{d\hat{w}^*_j(\xi)}{d\xi}\Big) = 0$$

$$(\xi = 1, \qquad \xi = R_0/r_0).$$

The expressions for the derivatives are rather cumbersome and are not presented here.

As the solutions of the homogeneous equation (18) are readily found, the particular solution can also be obtained by a famous technique [7] according to which the particular solution is found with the use of the Green function $G(\rho, r)$ for the initial equation. Then, the particular solution is written as

$$\hat{w}(r) = \int_{r_0}^{R_0} G(r, \rho) V(\rho)\, d\rho, \tag{38}$$

where $V(\rho)$ is the right side of the initial equation.

To determine the Green function, we use its representation in the form

$$G(r, \rho) = \sum_{i=1}^{4} \chi_i(\rho) \tilde{w}_i(r)$$

[$\tilde{w}_i(r)$ is the solution of the homogeneous equation], and then the functions $\chi_i(\rho)$ are found from a system of linear equations as functions of the known $\tilde{w}_i(\rho)$ and their derivatives:

$$\sum_{i=1}^{4} \chi_i(\rho) \frac{d^i}{d\rho^i} \tilde{w}_i(\rho) = \begin{cases} 0, & i \leqslant 3, \\ 1/x_0(\rho), & i = 4 \end{cases}$$

[$x_0(\rho)$ is the coefficient at the higher derivative].

In our case, $x_0(\rho) = \rho^2$ and $V(\rho) = (q_0/D)\rho^2[\Omega(\rho - \rho_1) - \Omega(\rho - \rho_2)]$. Being substituted into (38), the solution found for $G(r, \rho)$ yields the sought particular solution

$$\hat{w}(r) = \sum_{i=1}^{4} \tilde{w}_i(r) \int_{r_0}^{r} \chi_i(\rho) V(\rho)\, d\rho = \sum_{i=1}^{4} \tilde{w}_i(r) \int_{r_1}^{r} \chi_i(\rho) V(\rho)\, d\rho.$$

4. Bending State of the Shell. The differential equation that describes the bending state of the conical shell is written with respect to the radial displacement of the shell $w(r)$ and has the form of the Euler equation:

$$r^4 \frac{d^4 w}{dr^4} + 2r^3 \frac{d^3 w}{dr^3} - (1 + 2k^2 n^2) r^2 \frac{d^2 w}{dr^2} + (1 + 2k^2 n^2) r \frac{dw}{dr} + k^2 n^2 (k^2 n^2 - 4) = \frac{r^4}{D} q_0 \varphi_n \varphi(r). \tag{39}$$

In accordance with the asymptotic synthesis method, the stress–strain state is constructed for harmonics with numbers $n > n^*$, where n^* is found by the formula derived in [2, p. 284].

The sought solution of the equation is presented as the sum of the solution of the homogeneous equation $\tilde{w}(r)$ and the particular solution $\hat{w}(r)$:

$$w(r) = \tilde{w}(r) + \hat{w}(r). \tag{40}$$

Substituting the function $w(r)$ from (40) presented in the form of the series

$$w(r) = \sum_{-\infty}^{\infty} a_m r^m$$

into Eq. (39) with the zero right side, we obtain the equation for m:

$$P_4(m) = m(m-1)(m-2)(m-3) + 2m(m-1)(m-2) - (1 + 2k^2 n^2) m(m-1) + (1 + 2k^2 n^2) m + k^2 n^2 (k^2 n^2 - 4) = 0. \tag{41}$$

Obviously, the solution of the homogeneous equation can be written as

$$\tilde{w}(r) = C_1 r^{\lambda_1} + C_2 r^{\lambda_2} + C_3 r^{\lambda_3} + C_4 r^{\lambda_4},$$

where λ_j $(j = 1, 2, 3, 4)$ are the roots of Eq. (41) and C_j are arbitrary constants.

To find the solution of the heterogeneous equation $\hat{w}(r)$, we apply the Mellin transform to both sides of Eq. (39) with a prior substitution of the variable $r = r_0 \xi$ or introduce a new variable $\xi = r/r_0$:

$$\xi^4 \frac{d^4 w^*}{d\xi^4} + 2\xi^3 \frac{d^3 w^*}{d\xi^3} - (1 + 2k^2 n^2) \xi^2 \frac{d^2 w^*}{d\xi^2} + (1 + 2k^2 n^2) \xi \frac{dw^*}{d\xi} + k^2 n^2 (k^2 n^2 - 4) w^* = \frac{q_0 r_0^4}{D} \xi^4 \varphi_n \varphi(\xi). \tag{42}$$

Here $w^* = w^*(\xi) = w(r_0 \xi)$.

Then, applying the Mellin transform to both parts of Eq. (42), we obtain

$$P_4(p)\tilde{w}(p) = \tilde{g}(p), \tag{43}$$

where $\tilde{g}(p)$ is the Mellin transform for the function $g(\xi) = q_0 r_0^4(\varphi_n/D)\varphi(\xi)\xi^4 = b_n\xi^4\varphi(\xi)$:

$$\tilde{g}(p) = \int_0^\infty g(\xi)\xi^{p-1}\,d\xi = \int_{\xi_1}^{\xi_2} b_n\xi^{p+3}[\Omega(\xi-\xi_1) - \Omega(\xi-\xi_2)]\,d\xi = \frac{b_n(\xi_2^{p+4} - \xi_1^{p+4})}{p+4}.$$

Here $\xi_j = r/r_0$ and $\Omega(\xi)$ is the Heaviside function.

It follows from equality (43) that

$$\tilde{w}(p) = \tilde{g}(p)/P_4(p).$$

Let us now prove that two roots λ_j of Eq. (41) lie in the right half-plane of the complex plane and two other roots lie in the left half-plane. The proof directly follows from the Vyshnegradskii–Nyquist theorem [11] formulated below.

Theorem 1. *The number of roots of the function $f(z,\zeta) = P_1(z)\zeta - P_2(z)$ in the left half-plane for $\zeta = \zeta_0$ is*

$$K(\zeta_0) = (2\pi)^{-1}\Delta_\Gamma \arg(\zeta_0 - \zeta) + m + k_1, \tag{44}$$

where $P_1(z)$ and $P_2(z)$ are polynomials ($z = x + iy$), Γ is the hodograph of the function $\zeta(z) = P_2(z)/P_1(z)$ for y increasing from $-\infty$ to $+\infty$ and $x = 0$, k_1 is the number of roots of the polynomial $P_1(z)$ in the left half-plane, $m = n_2 - n_1$ ($n_2 > n_1$), $m = 0$ ($n_2 < n_1$), and n_1 and n_2 are the powers of the polynomials $P_1(z)$ and $P_2(z)$, respectively.

We assume that

$$P_1(z) = z^4 - (1+2k^2n^2)z^2 + k^2n^2(k^2n^2-4), \qquad P_2(z) = 4z^3 + (1-2k^2n^2)z,$$

$$\zeta(z)(z=iy) = \frac{P_2(z)}{P_1(z)}(z=iy) = -\frac{4y^3 + (2k^2n^2-1)y}{y^4 + (1-2k^2n^2)y^2 + k^2n^2(k^2n^2-4)}.$$

Hence, the hodograph $\Delta_\Gamma \arg(\zeta_0 - \zeta)$, where $\zeta_0 = 1$, equals zero, as is easily seen.

As two roots of the biquadratic equation $P_1(z) = 0$ lie in the left half-plane, i.e., $k_1 = 2$ ($m = 0$), we find from Eq. (44) that $k_1 = 2$, i.e., two roots λ_j lie in the left half-plane, Q.E.D.

We represent the polynomial $P_4(p)$ in the form $P_4(p) = P_0(p)P_1(p)$, where the roots of the polynomial $P_0(p)$ lie in the left half-plane (λ_1, λ_2), and the roots of the polynomial $P_1(p)$ lie in the right half-plane (λ_3, λ_4).

Applying the Mellin transform to both sides of Eq. (42), we obtain

$$w^*(\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\tilde{g}(p)}{P_4(p)}\,\xi^{-p}\,dp = \frac{b_n}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(\xi_2^{p+4} - \xi_1^{p+4})\xi^{-p}\,dp}{P_4(p)(p+4)} = \frac{b_n}{2\pi i}\int_{-i\infty}^{i\infty} \frac{(\xi_2^{p+4} - \xi_1^{p+4})\xi^{-p}}{(p+4)P_0(p)P_1(p)}\,dp = [f(\xi,\xi_2) - f(\xi,\xi_1)].$$

We calculate the integral

$$f(\xi,\xi_j) = \frac{b_n}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\xi_j^{p+4}\xi^{-p}}{(p+4)P_0(p)P_1(p)}\,dp = \frac{b_n\xi_j^4}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp(p\ln(\xi_j/\xi))}{(p+4)P_0(p)P_1(p)}\,dp \quad (j = 1,2).$$

The integrand satisfies the conditions of Jordan's lemma in the left half-plane of the complex variable p for $\xi < \xi_j$ and in the right half-plane for $\xi > \xi_j$. The integrand has three simple poles in the left half-plane ($p_1 = -4$, $p_2 = \lambda_1$, and $p_3 = \lambda_2$) and two simple poles in the right half-plane ($p_4 = \lambda_3$ and $p_5 = \lambda_4$).

Thus, we obtain

$$f(\xi,\xi_j) = b_n\xi_j^4\Big[\frac{(\xi/\xi_j)^4}{P_4(-4)} + \frac{(\xi/\xi_j)^{\lambda_1}}{(\lambda_1+4)(\lambda_1-\lambda_2)P_1(\lambda_1)} + \frac{(\xi/\xi_j)^{\lambda_2}}{(\lambda_2+4)(\lambda_2-\lambda_1)P_1(\lambda_2)}\Big], \quad \xi < \xi_j,$$

$$f(\xi,\xi_j) = -b_n\xi_j^4\Big[\frac{(\xi_j/\xi)^{\lambda_3}}{P_4(\lambda_3)P_0(\lambda_3)(\lambda_3-\lambda_4)} + \frac{(\xi_j/\xi)^{\lambda_4}}{P_4(\lambda_4)P_0(\lambda_4)(\lambda_4-\lambda_3)}\Big], \qquad \xi > \xi_j.$$

We write the sought solution in the form

$$w(r) = w^*(r_0\xi) = w^*(\xi) = f(\xi,\xi_2) - f(\xi,\xi_1).$$

To calculate the forces and moments, we have to find expressions for the first three derivatives of the radial displacement:

$$\frac{dw}{dr} = \frac{dw^*}{r_0\,d\xi}, \qquad \frac{d^2w}{dr^2} = \frac{d^2w^*}{r_0^2\,d\xi^2}, \qquad \frac{d^3w}{dr^3} = \frac{d^3w^*}{r_0^3\,dr^3},$$

$$\frac{d^{(q)}w^*}{d\xi^{(q)}} = \frac{d^{(q)}f(\xi,\xi_2)}{d\xi^{(q)}} - \frac{d^{(q)}f(\xi,\xi_1)}{d\xi^{(q)}} \qquad (q = 1,2,3,\dots).$$

The derivatives of the function $f(\xi,\xi_j)$ are calculated by the formulas

$$f'(\xi,\xi_j) = b_n\xi_j^4\Big[\frac{(4/\xi_j)(\xi/\xi_j)^3}{P_4(-4)} + \frac{(\lambda_1/\xi_j)(\xi/\xi_j)^{\lambda_1-1}}{(\lambda_1+4)(\lambda_1-\lambda_)P_1(\lambda_1)} + \frac{(\lambda_2/\xi_j)(\xi/\xi_j)^{\lambda_2-1}}{(\lambda_2+4)(\lambda_1-\lambda_2)P_1(\lambda_2)}\Big], \quad \xi < \xi_j,$$

$$f'(\xi,\xi_j) = b_n\xi_j^4\Big[\frac{(\lambda_3/\xi_j)(\xi_j/\xi)^{\lambda_3+1}}{P_4(\lambda_3)P_0(\lambda_3)(\lambda_3-\lambda_4)} + \frac{(\lambda_4/\xi_j)(\xi_j/\xi)^{\lambda_4+1}}{P_4(\lambda_4)P_0(\lambda_4)(\lambda_4-\lambda_3)}\Big], \quad \xi > \xi_j;$$

$$f''(\xi,\xi_j) = b_n\xi_j^4\Big[\frac{(12/\xi_j^2)(\xi/\xi_j)^2}{P_4(-4)} + \frac{(\lambda_1(\lambda_1-1)/\xi_j^2)(\xi/\xi_j)^{\lambda_1-2}}{(\lambda_1+4)(\lambda_1-\lambda_2)P_1(\lambda_1)}$$

$$+ \frac{(\lambda_2(\lambda_2-1)/\xi_j^2)(\xi/\xi_j)^{\lambda_2-2}}{(\lambda_2+4)(\lambda_2-\lambda_1)P_1(\lambda_2)}\Big], \qquad \xi < \xi_j,$$

$$f''(\xi,\xi_j) = -b_n\xi_j^4\Big[\frac{(\lambda_3(\lambda_3+1)/\xi^2)(\xi_j/\xi)^{\lambda_3+2}}{P_4(\lambda_3)P_0(\lambda_3)(\lambda_3-\lambda_4)} + \frac{(\lambda_4(\lambda_4+1)/\xi^2)(\xi_j/\xi)^{\lambda_4+2}}{(\lambda_4+4)(\lambda_4-\lambda_3)P_1(\lambda_4)}\Big], \quad \xi > \xi_j;$$

$$f'''(\xi,\xi_j) = b_n\xi_j^4\Big[\frac{(24/\xi_j^3)(\xi/\xi_j)}{P_4(-4)} + \frac{(\lambda_1(\lambda_1-1)(\lambda_1-2)/\xi_j^3)(\xi/\xi_j)^{\lambda_1-3}}{(\lambda_1+4)(\lambda_1-\lambda_2)P_1(\lambda_1)}$$

$$+ \frac{(\lambda_2(\lambda_2-1)(\lambda_2-2)/\xi_j^3)(\xi/\xi_j)^{\lambda_2-3}}{(\lambda_2+4)(\lambda_2-\lambda_1)P_1(\lambda_2)}\Big], \quad \xi < \xi_j,$$

$$f'''(\xi,\xi_j) = b_n\xi_j^4\Big[\frac{(\lambda_3(\lambda_3+1)(\lambda_3+2)/\xi_j^3)(\xi_j/\xi)^{\lambda_3+3}}{P_4(\lambda_3)P_0(\lambda_3)(\lambda_3-\lambda_4)}$$

$$+ \frac{(\lambda_4(\lambda_4+1)(\lambda_4+2)/\xi_j^3)(\xi_j/\xi)^{\lambda_4+3}}{(\lambda_4+4)(\lambda_4-\lambda_3)P_1(\lambda_4)}\Big], \quad \xi > \xi_j.$$

The general solution has the form

$$w(r) = \tilde{w}(r) + \hat{w}(r) = \sum_{i=1}^{4} C_s r^{\lambda_i} + \hat{w}(r). \tag{45}$$

For a semi-infinite conical shell, we need to set $C_3 = C_4 = 0$. For a finite-length shell with free edges, the boundary conditions take the form

$$G_1(r) = \Big[\frac{d^2w}{dr^2} + \nu\Big(\frac{1}{r}\,\frac{dw}{dr} - \frac{k^2n^2}{r^2}\,w\Big)\Big]_{\substack{r=r_0\\ r=R_0}} = 0,$$

$$Q_1(r) = \Big[\frac{d}{dr}\Big(\frac{d^2w}{dr^2} + \frac{1}{r}\,\frac{dw}{dr} - k^2n^2\,\frac{w}{r^2}\Big)\Big]_{\substack{r=r_0\\ r=R_0}} = 0. \tag{46}$$

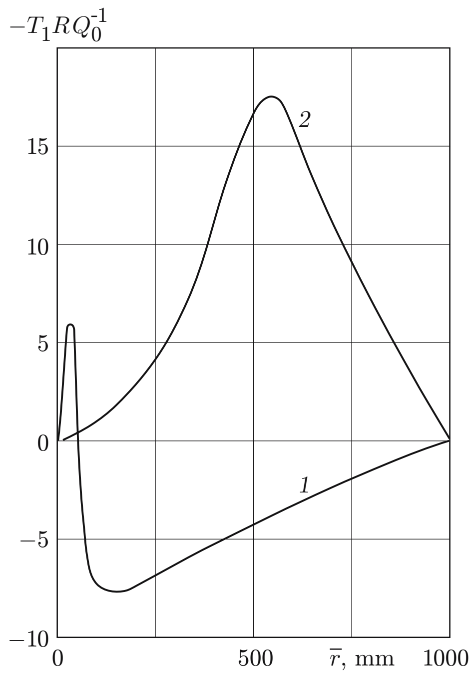

Fig. 2

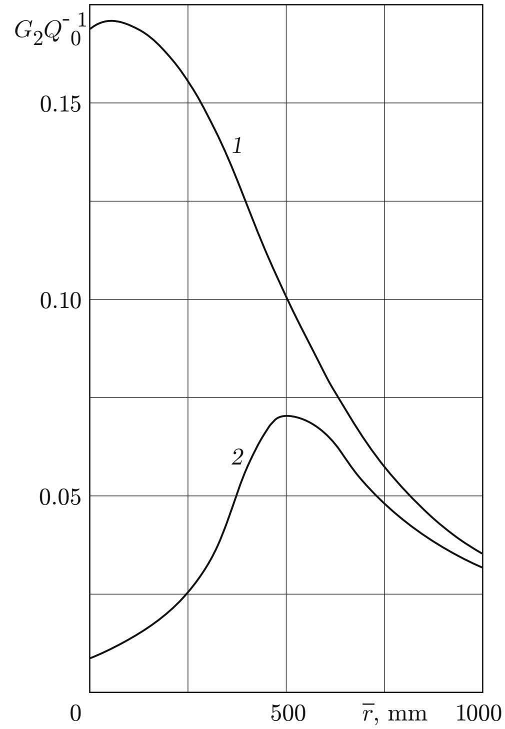

Fig. 3

Substituting expressions (45) into (46), we obtain a system of algebraic equations with respect to the constants C_s $(s = 1, 2, 3, 4)$

$$\begin{aligned}
&\sum_{j=1}^{4}[\lambda_j(\lambda_j - 1) + \nu(\lambda_j - k^2n^2)]r_0^{\lambda_j-2}C_j = g_1,\\
&\sum_{j=1}^{4}[\lambda_j^2(\lambda_j - 2) - k^2n^2\lambda_j + 2k^2n^2]r_0^{\lambda_j-3}C_j = g_2,\\
&\sum_{j=1}^{4}[\lambda_j(\lambda_j - 1) + \nu(\lambda_j - k^2n^2)]R_0^{\lambda_j-2}C_j = g_3,\\
&\sum_{j=1}^{4}[\lambda_j^2(\lambda_j - 2) - k^2n^2\lambda_j + 2k^2n^2]R_0^{\lambda_j-3}C_j = g_4
\end{aligned} \tag{47}$$

with the following expressions for the right sides of Eqs. (47):

$$g_1 = -\frac{1}{r_0^2}\Big[\frac{d^2w^*}{d\xi^2} + \nu\Big(\frac{1}{\xi}\,\frac{dw^*}{d\xi} - \frac{k^2n^2}{\xi^2}\,w^*\Big)\Big]\Big|_{\substack{\xi=1\\ r=r_0}},$$

$$g_2 = -\frac{1}{r_0^3}\Big[\frac{d}{d\xi}\Big(\frac{d^2w^*}{d\xi^2} + \frac{1}{\xi}\,\frac{dw^*}{d\xi} - k^2n^2\,\frac{w^*}{\xi^2}\Big)\Big]\Big|_{\substack{\xi=1\\ r=r_0}},$$

$$g_3 = -\frac{1}{r_0^2}\Big[\frac{d^2w^*}{d\xi^2} + \nu\Big(\frac{1}{\xi}\,\frac{dw^*}{d\xi} - k^2n^2\,\frac{w^*}{\xi^2}\Big)\Big]\Big|_{\substack{\xi=R_0/r_0\\ r=R_0}},$$

$$g_4 = -\frac{1}{r_0^3}\Big[\frac{d}{d\xi}\Big(\frac{d^2w^*}{d\xi^2} + \frac{1}{\xi}\,\frac{dw^*}{d\xi} - k^2n^2\,\frac{w^*}{\xi^2}\Big)\Big]\Big|_{\substack{\xi=R_0/r_0\\ r=R_0}}.$$

Thus, the solution of the problem posed can be assumed to be constructed.

5. Calculation Example. As an example, we consider a conical shell with free edges ($r_0 = 585$ mm and $R_0 = 1585$ mm) under the action of two ($k = 2$) inward-directed local radial loads Q_0 whose applications points are located on one diameter. For regions where the load is uniformly distributed with intensity q_0, we have $r_1 = 598$ mm, $r_2 = 623$ mm, and $\beta_0 = 0.125$. The diameters of the end faces of the shell are 184 and 556 mm, and the shell thickness is 1 mm.

The calculation results are plotted in Figs. 2 and 3. Figure 2 shows the longitudinal force $T_1 R Q_0^{-1}$ (R is the shell radius for $r = r_0$; $\beta = 0$) as a function of the longitudinal coordinate $\bar{r} = r - r_0$ (curve 1), and Fig. 3 shows the bending moment $G_2 Q_0^{-1}$ versus $\bar{r}$ (curve 1). Curves 2 in Figs. 2 and 3 refer to the same dependences in the case the load is applied in the middle of the shell. Note that the axial component arising in the shell owing to the normal loads Q_0 applied is balanced at the larger end face by an infinitesimal distributed stress.

For a hinge-supported conical shell under the action of concentrated forces, the radial displacement was compared with the results of [2]. The difference in the maximum values of displacements is within 5%.

This work was partly supported by the International Science Foundation (Grant No. N2J300).

REFERENCES

1. I. F. Obraztsov, B. V. Nerubailo, and I. V. Andrianov, *Asymptotic Methods in Mechanical Engineering of Thin-Walled Structures* [in Russian], Mashinostroenie, Moscow (1991).
2. B. V. Nerubailo, "Solution of differential equations for conical shells with a singular right-side part," *J. Appl. Mech. Tech. Phys.*, **37**, No. 2, 283–286 (1996).
3. A. D. Kovalenko, Ya. M. Grigorenko, and L. A. Il'in, *Theory of Thin Conical Shells and its Application in Mechanical Engineering* [in Russian], Izd. Akad. Nauk. Ukr. SSR, Kiev (1963).
4. B. V. Nerubailo, *Local Problems of Strength of Cylindrical Shells* [in Russian], Mashinostroenie, Moscow (1983).
5. S. N. Kan, *Engineering Mechanics of Shells* [in Russian], Mashinostroenie, Moscow (1966).
6. W. Kecs and P. Teodorescu, *Introduction into the Theory of Generalized Functions with Engineering Applications* [Russian translation], Mir, Moscow (1978).
7. J. S. Ufljand, *Integral Transformations in Problems of the Elasticity Theory* [Russian translation], Nauka, Leningrad (1968).
8. E. Madelung, *Mathematical Apparatus of Physics* [Russian translation], Nauka, Moscow (1968).
9. J. Colle, *Perturbations Methods in Applied Mathematics*, Blaisdell Publishing Company, Waltham, USA (1968).
10. V. Vasow, *Asymptotic Expansions for Ordinary Differential Equations*, New York (1965).
11. M. A. Lavrent'ev and V. V. Shabat, *Methods of the Theory of Functions of Complex Variables* [in Russian], Nauka, Moscow (1973).